


ON FINITE ORDER INVARIANTS OF TRIPLE POINTS FREE PLANE CURVES

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To D. B. Fuchs with love

ABSTRACT. We describe some regular techniques of calculating finite-order invariants of triple points free smooth plane curves $S^1 \rightarrow \mathbf{R}^2$. They are a direct analog of similar techniques for knot invariants and are based on the calculus of *triangular diagrams* and *connected hypergraphs* in the same way as the calculation of knot invariants is based on the study of chord diagrams and connected graphs.

E.g., the simplest such invariant is of order 4 and corresponds to the triangular diagram  in the same way as the simplest knot invariant (of order 2) corresponds to the 2-chord diagram \oplus . Also, following V. I. Arnold and other authors we consider invariants of *immersed* triple points free curves and describe similar techniques also for this problem, and, more generally, for the calculation of homology groups of the space of immersed plane curves without points of multiplicity $\geq k$ for any $k \geq 3$.

INTRODUCTION

The intensive study of invariants of generic immersions $S^1 \rightarrow \mathbf{R}^2$ was started by V. I. Arnold in [5] and continued in [6], [7], [38], [1], [30], [27], [25], [24], etc.

The most interesting invariant of such objects, the *strangeness*, is in fact an invariant of triple points free immersions $S^1 \rightarrow \mathbf{R}^2$ (with allowed self-tangencies).

Almost simultaneously, [35], [33], I considered the *ornaments*, i.e. collections of plane curves (maybe with singularities) without intersections of three different components, and developed regular techniques for calculating their invariants. The present work is the (promised in

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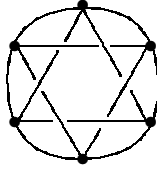


FIGURE 1. Unique invariant of order 4 for doodles

[35]) substitution of these methods into the theory of triple points free plane curves.

Below we describe a natural filtration of invariants by their *orders*, and a regular method of calculating all invariants of all finite orders for triple points free plane curves. Following the idea of [3], we reduce the study of invariants (and other cohomology classes of the space of generic objects) to that of the homology groups of the complementary *discriminant set* of objects with forbidden singularities (i.e., in our case, of curves with triple points). The more technical tools of this method are the simplicial resolutions of discriminants (see [31]) and the (arising from them) calculus of *triangular diagrams* and *connected hypergraphs*, which are analogs of chord diagrams and connected graphs arising in the theory of finite-order knot invariants.

The simplest such invariants are described in following two theorems.

First, as was proposed in [35] (see Problem 2 of § 9 there), we consider the space of all plane curves $\phi : S^1 \rightarrow \mathbf{R}^2$ having no triple points and no singularities obtained as degenerations of triple points (i.e., either the double points at which one of two local branches has a singular point with $\phi' = 0$, or the points at which $\phi' = \phi'' = 0$.) The problem of classifying such objects (called the *doodles*) is in the same relation with the classification of ornaments, in which the isotopy classification of links is with the homotopy classification.

In this setting, the curves “0” and “8” become equivalent, and the strangeness fails to be an invariant of such objects; this is an analog of the fact that the trivial chord diagram \ominus does not define a knot invariant.

Theorem 1. *There are no invariants of doodles of orders 1, 2 or 3, and there is exactly one invariant of order 4.*

This invariant is depicted by the triangular diagram shown in Fig. 1.

(This diagram is an adequate analog of the chord diagram \oplus defining the first nontrivial knot invariant: they both are simplest diagrams of corresponding kinds, not containing elements with neighboring vertices.)

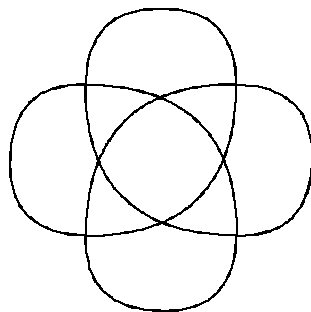


FIGURE 2. A plane curve not equivalent to the circle in the space of singular triple points free curves (after A. B. Merkov)

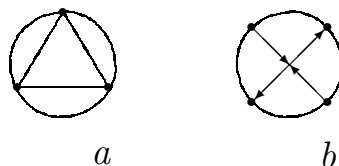


FIGURE 3. Notation for invariants of order 2 and 3 of immersed curves

This invariant proves, in particular, that the curve of Fig. 2 (discovered previously by A. B. Merkov) is not equivalent to a circle.

Further, following [5], let us consider the immersed curves in \mathbf{R}^2 .

Theorem 2. *There are only the following invariants of orders ≤ 4 of triple points free plane **immersed** curves $S^1 \rightarrow \mathbf{R}^2$:*

- 1) *no invariants of order 1;*
- 2) *one invariant of order 2 (the Arnold's strangeness; by some reasons we denote this invariant by the simplest "triangular diagram" of Fig. 3a;*
- 3) *one more invariant of order 3 (its natural notation see in Fig. 3b;*
- 4) *five more invariants of order 4 (they are described in Fig. 4).*

Our methods allow us to calculate also some higher-dimensional cohomology classes of spaces of k -points free plane curves (both immersed or just C^∞ -parametrized) for any $k \geq 3$.

E.g., let $k = 4$. The set $\Sigma 4$ of all curves with 4-fold selfintersections has codimension 2 in the space of all plane curves, thus the first interesting problem is the calculation of the 1-dimensional cohomology

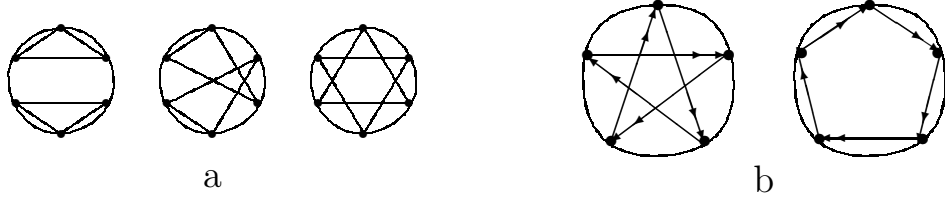


FIGURE 4. Notation for invariants of order 4

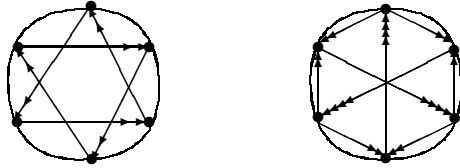


FIGURE 5. Two generators of order 5 of the 1-cohomology group of the space of plane immersed curves without 4-fold points

group of the complementary space of immersed plane curves without such points¹.

This cohomology group $H^1(Imm(S^1, \mathbf{R}^2) \setminus \Sigma_4)$ also has a natural filtration, so that the *orders* of (some) its elements are well-defined.

Theorem 3. *For any connected component of the space of immersions $S^1 \rightarrow \mathbf{R}^2$, the first few groups \mathcal{F}_d of order d 1-dimensional integer cohomology classes of the space of four-points free immersions lying in this component are as follows: $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = 0$, $\mathcal{F}_5 \simeq \mathbf{Z}^2$.*

However, if we calculate the \mathbf{Z}_2 -cohomology, then we have $\mathcal{F}_3 \simeq \mathbf{Z}_2$, $\mathcal{F}_4/\mathcal{F}_3 = 0$, $\mathcal{F}_5/\mathcal{F}_4 \simeq \mathbf{Z}_2^2$, and if we calculate the \mathbf{Z}_5 -cohomology, then $\mathcal{F}_3 = 0$, $\mathcal{F}_4 \simeq \mathbf{Z}_5$, $\mathcal{F}_5/\mathcal{F}_4 \simeq \mathbf{Z}_5^2$.

Two generators of the group $\mathcal{F}_5/\mathcal{F}_4$ (with integer coefficients) are naturally depicted by two chains shown in Fig. 5.

Many invariants of immersions from Theorem 1 have elementary description. Namely, the sum of three generators from Fig. 4a is equal to the square of the strangeness. Moreover, strangeness itself, the unique invariant of order 3 and the sum of two invariants of order 4 shown in Fig. 4b, are "index-type invariants," see § 1.3 below, thus initiating an infinite series of finite-order invariants (one in each order) of this sort.

¹The problem of calculating such homology groups, posed by V. I. Arnold (see [8], problem 1996-2) forced me to write this paper

IMPORTANT NOTE. Our notion of the order of invariants differs from the one used in [5]—[7], [27], [30] etc. Any invariant of finite order k in the sense of our work is also of order $\leq [k/2]$ in the sense of these works; the converse is false very much.

There are (among others) three equivalent definitions of finite order invariants of knots in \mathbf{R}^3 : 1) the "geometrical", in terms of resolved discriminants and their filtrations, 2) the "axiomatic", in terms of finite differences of knot diagrams, and 3) the "combinatorial" (developed in [26]) in terms of homomorphisms of chord diagrams. The equivalence of two first definitions was clear from the very beginning, their equivalence to the third one is a nontrivial fact, conjectured by M. Polyak and O. Viro and proved by M. Goussarov.

There is a wide class of objects (including knots, ornaments, and doodles), whose invariants can be calculated by the methods, developed in [32], [35], i.e. in the terms of resolved discriminants, thus leading to the "geometrical" definition of finite-order invariants. An "axiomatic" elementary reformulation of the resulting notion in our present situation also exists, but it is not a straightforward translation of that from [32], see § 2 below. I believe that it will lead to the most interesting algebraic structures, reflecting the rich geometric structures staying behind it.

The "combinatorial" definition and related aspects of the same invariants of ornaments and doodles are introduced and investigated by A. B. Merkov, [21]—[24] as a far generalization of the index-type invariants from [35].

In particular, he proved that these invariants distinguish any two nonequivalent collections of (arbitrarily many) plane curves without triple intersections or selfintersections. However I believe that the techniques of the present work allow to calculate all such invariants in the most direct and regular way.

I thank very much A. B. Merkov for numerous consultations and other multiform help.

1. ELEMENTARY THEORY

This and the next sections are almost exact analogues of §§ 1, 2 from [35].

1.1. First definitions and Reidemeister moves.

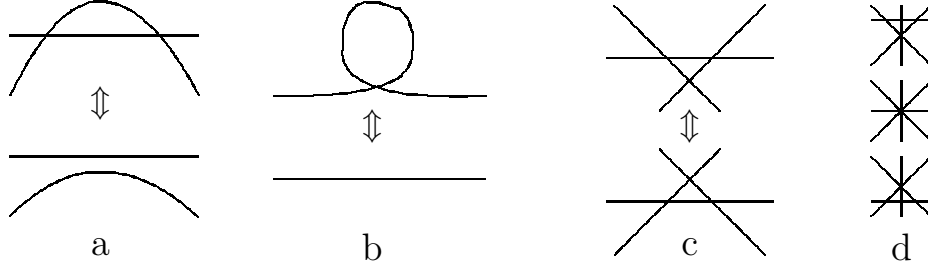


FIGURE 6. Standard moves of quasidoodles

Definition 1. A *doodle* is a C^∞ -map $\phi : S^1 \rightarrow \mathbf{R}^2$ such that for none three different points $x, y, z \in S^1$ one the following conditions holds ²:

$$(1.1) \quad \phi(x) = \phi(y) = \phi(z)$$

$$(1.2) \quad \phi'(x) = 0, \phi(x) = \phi(y)$$

$$(1.3) \quad \phi'(x) = \phi''(x) = 0.$$

An *I-doodle* (i.e. immersed doodle) is a doodle which is an immersion (i.e. a map ϕ without degenerations of two types (1.1) and

$$(1.4) \quad \phi'(x) = 0).$$

Two doodles (respectively, I-doodles) are *equivalent* if there is a continuous family of doodles (I-doodles) connecting them. An *invariant* of doodles or I-doodles is any function on the space of these objects, taking equal values at equivalent objects.

A doodle is *regular* if it is an immersion having only transverse double points.

Proposition 1. *Any equivalence class of doodles or I-doodles contains regular doodles. Two regular doodles define equivalent doodles (respectively, I-doodles) if and only if they can be transformed one into the other by a finite sequence of isotopies of \mathbf{R}^2 (which do not change the topological picture of the image of the doodle), and of local moves shown in Fig. 6a, b (respectively, 6a only).*

In other words, the move of Fig. 6c is prohibited in the classification of doodles, and both 6b, 6c in the case of I-doodles.

The proof of this proposition is trivial.

²In a more general theory, see [17], [24], this object is called an 1-doodle. We consider here only such one-component doodles and call them simply *doodles*.

Definition 2. A *quasidoodle* is any C^∞ -map $S^1 \rightarrow \mathbf{R}^2$. The space of all such maps is denoted by \mathcal{K} . The space $Imm(S^1, \mathbf{R}^2)$ of all immersions $S^1 \rightarrow \mathbf{R}^2$ will be denoted by IK . The *discriminant* (respectively, I-discriminant) $\Sigma \subset \mathcal{K}$ (respectively, $I\Sigma \subset IK$) is the set of all maps from this space for which one of prohibited conditions (1.1)—(1.3) (respectively, (1.1)) is satisfied.

Proposition 2. *The set Σ is a closed subvariety of codimension 1 in \mathcal{K} . The set of all maps, for which only (1.1) is satisfied, is dense in Σ , and the maps satisfying (1.2) or (1.3) lie in its closure. The set of quasidoodles having no degenerations of types (1.1) and (1.2) but with allowed degenerations of type (1.3) is path-connected in \mathcal{K} .*

The proof of the last assertion essentially coincides with that of the fact that *all* embeddings $S^1 \rightarrow \mathbf{R}^3$ (maybe not regular) form a path-connected subset in $C^\infty(S^1, \mathbf{R}^3)$, see e.g. [13]. All other statements of the proposition are elementary.

1.2. On Arnold's invariants of immersed plane curves. In [5] V. I. Arnold introduced three invariants of generic immersed plane curves. One of them is the *strangeness*, defined as the linking number in IK with the suitably (co)oriented variety $I\Sigma \subset IK$. The coorientation of this variety, participating in this construction, will be specified in § 1.4.

1.3. Index-type invariants of I-doodles. Let us fix an orientation of the plane \mathbf{R}^2 .

Recall that any closed oriented immersed curve c in \mathbf{R}^2 defines an integer-valued function ind_c on its complement: for any point t of the complement, $ind_c(t)$ equals the (counterclockwise) rotation number of the vector (t, x) when x runs one time along c .

Consider a regular I-doodle $\phi : S^1 \rightarrow \mathbf{R}^2$. To any self-intersection point x of the curve $c = \phi(S^1)$ assign its index $i(x)$ equal to the arithmetical mean of four values of ind_c in four neighboring components of $\mathbf{R}^2 \setminus c$, see Fig. 7. Let us fix a regular (not intersection) point $*$ in c and define its index $i(*)$ as the greatest value of ind_c in two neighboring domains of the complement of c . For any selfintersection point x consider the frame in it, formed by (oriented) tangent vectors to c . These vectors are ordered in correspondence with the number of visits of c after leaving the point $*$. Define the *sign* $\sigma(x)$ of the point x as the sign of the orientation of this ordered frame.

For any integer i and natural β , denote by $\frac{i}{\beta}$ the number $i(i-1)\cdots(i-\beta+1)/\beta!$, cf. [22], [35]. It is easy to see that this number is always integer.

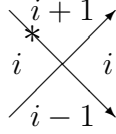


FIGURE 7.

For any natural β , define the β -th moment $M(\beta)$ of the regular doodle by the equality

$$(1.5) \quad M(\beta) = \sum_x \sigma(x) \binom{i(x)}{\beta} + 2 \binom{i(*)}{\beta+1}.$$

Proposition 3 (cf. [28], [35]). *All numbers $M(\beta)$, $\beta = 1, 2, \dots$, are invariants of I-doodles, in particular do not depend on the choice of the distinguished point $*$. The first of them, $M(1)$, is the Arnold's strangeness.*

Remark 1. In the very similar case of ornaments such invariants were introduced in [35], § 1.4, as first nontrivial examples of finite-order invariants, see also [22]. In [30] similar expressions appeared as invariants of one-component *long curves*, i.e. essentially of curves with a fixed nonsingular point $*$. The formulae for all these invariants contained only the terms similar to the first term of the right-hand part of (1.5). Finally, A. Shumakovich [28] introduced a correcting second term and obtained invariants independent on the choice of this point: these his invariants coincide with (1.5) up to a linear transformation with rational coefficients. (The simplest version of this correcting term, corresponding to the case $\beta = 1$ and providing a combinatorial expression for the Arnold's strangeness, appeared previously in [27].) Numerous more general combinatorial expressions for invariants of doodles, ornaments, I-doodles, etc. were given in [21]–[24]. It seems likely that the method described below (see also [35], [22]) is the most universal algorithm for guessing such expressions: first one calculates several first elements of a spectral sequence converging to the group of all invariants of finite degree, and then finds an elementary interpretation for them; cf. also [32], [33].

1.4. Coorientation of the discriminant. The discriminant set Σ has a natural (co)orientation in its regular points: if we go along a generic path in the space \mathcal{K} and traverse the discriminant, doing the local surgery shown in Fig. 6c, then there is an invariant way to say, which one of these two resolved pictures lies on the positive side of

the discriminant, and which on the negative. There are numerous equivalent definitions of this coorientation, see e.g. [5], [27], [22]. One of them can be formulated as follows: we consider the sum like (1.5) with $\beta = 1$, but with summation only over 3 points participating in the surgery. The positive (negative) side of discriminant is that with the greater (smaller) value of this sum.

This coorientation is well defined even if the curve has forbidden multiple (or, in the theory of I-doodles, forbidden singular) points far away from the location of the surgery: in fact, this is a coorientation of the locally irreducible branch of the discriminant set.

2. ELEMENTARY DEFINITION OF THE ORDER OF INVARIANTS

2.1. Degeneration modes and characteristic numbers. The orders of invariants of doodles and I-doodles will be defined in the same way.

Definition 3. Suppose that j is a natural number, $j \geq 2$. A *degree j standard singularity* of doodles is a pair of the form $\{\text{a quasidoodle } \phi : S^1 \rightarrow \mathbf{R}^2; \text{ a point } x \in \mathbf{R}^2\}$ such that $\phi^{-1}(x)$ consists of exactly $j+1$ points z_1, \dots, z_{j+1} , the map ϕ close to all these points is an immersion, and the corresponding $j+1$ local branches of the curve $\phi(S^1)$ are pairwise nontangent at x . A quasidoodle is called a *regular quasidoodle of complexity i* , if it is an immersion, all its forbidden points (i.e. the points, at which at least three different components meet) are standard singular points, and the sum of degrees of these singularities is equal to i .

Any regular quasidoodle can be obtained from regular doodles by a sequence of elementary degenerations. Namely, first we move along a generic path in the space \mathcal{K} , up to the first instant when some three points of $\phi(S^1)$ meet at the same point, forming a regular singularity of degree 2 (we do not watch the surgeries shown in Figs. 6a and 6b). Then we consider the vector subspace in \mathcal{K} , consisting of maps gluing together these three points of S^1 , and go along a generic path in it; at some instant either another triple point occurs or a fourth branch joins these three. Again, we fix the smaller subspace, consisting of maps gluing together all the same points, and move inside it. On the third step a point of multiplicity 5 can occur, or two points of multiplicities 4 and 3, or 3 points of multiplicity 3, etc. (In this case we do not watch also the nonessential local moves like the one shown in Fig. 8.) At the last step we get our quasidoodle.

Any such sequence of paths is called the *degeneration process* of our regular quasidoodle.

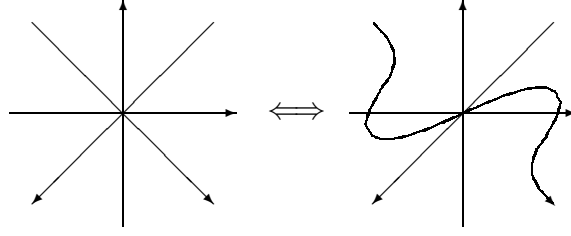


FIGURE 8. Nonessential move of a complicated multiple point

Examples. If our quasidoodle has only one triple point, then it can be obtained by two essentially different processes, corresponding to two its resolutions shown in Fig. 6c.

A quasidoodle with one singular point of multiplicity 4 has $16 = 4 \times 2 \times 2$ essentially different degeneration processes: at the first step any 3 of 4 points can meet at the same point of \mathbf{R}^2 (and this can be done in two different ways, see Fig. 6c), and on the second the fourth point joins them in one of two different ways shown in Fig. 6d.

If our quasidoodle has one more point of multiplicity 3, then there are $96 = 4 \times 3 \times 2^3$ different degeneration processes: the points of the second group can meet before, after, or between of two steps of degeneration of the first group.

Any degeneration process of a regular quasidoodle and any invariant of doodles (or I-doodles) defines a *characteristic number*, cf. § 2 of [35]. Namely, if we have a quasidoodle of some complexity j and a degeneration process DP of it, then there are exactly two quasidoodles of smaller complexity, whose degeneration processes coincide with DP without its last step. If at this last step some new group of multiplicity 3 occurs, then these are two resolutions of this group shown in Fig. 6c; if at this step one branch of $\phi(S^1)$ joined an existing group of multiplicity ≥ 3 , then we can move this branch to exactly two sides from this point, see Fig. 6d.

In all cases these two resolutions are ordered, i.e. one of them can be called positive and the other negative. In the first case this order is described in § 1.4.

In the second we define the index of a multiple point as the arithmetical mean of indices of points from all neighboring components of the complement of $\phi(S^1)$, and call *positive* the side for which the close point of multiplicity ≥ 3 has greater index with respect to the curve.

The characteristic number, which an invariant defines at the pair {a regular quasidoodle, some its degeneration process} is equal to the difference of similar numbers at two corresponding one-step resolutions

(at the positive minus at the negative one) supplied with the same degeneration process less its last step. (For these easier singularities these characteristic numbers are well defined by the inductive conjecture.)

Definition 4. An invariant is *of order j* if for any quasidoodle, whose complexity is greater than j , and any its degeneration process the corresponding characteristic number is equal to 0.

Proposition 4 (cf. Theorem 4 in [35]). *Any index-type invariant $M(\beta)$ from § 1.3 is of order $\beta + 1$. \square*

A wide class of invariants, generalizing these from § 1.3, was constructed by A. B. Merkov, see [22]. A further generalization of these invariants, [24], classifies all doodles up to equivalence; in particular it is as strong as entire space of all finite-order invariants.

2.2. Coding and calculation of finite-order invariants. Of course, the characteristic numbers of a degeneration process (and hence also the notion of the order) depend not of its geometrical realization, but only of some discrete data related with it, such as the combinatorial type of the set of points in S^1 pasted together at different its steps. Let us describe these data.

Let A be a finite series of integer numbers $A = (a_1, a_2, \dots, a_m)$, all of which are ≥ 3 . Denote by $|A|$ the number $a_1 + \dots + a_m$, and by $\#A$ the number of elements a_i of the series A (denoted in the previous line by m).

Definition 5. An A -configuration is a collection of $|A|$ pairwise different points in S^1 divided into groups of cardinalities $a_1, \dots, a_{\#A}$. Two A -configurations are *equivalent* if they can be transformed one into the other by an orientation-preserving diffeomorphism of S^1 . A quasidoodle $\phi : S^1 \rightarrow \mathbf{R}^2$ *respects* an A -configuration if it sends any of corresponding $\#A$ groups of points into one point in \mathbf{R}^2 . ϕ *strictly respects* the A -configuration if, moreover, all these $\#A$ points in \mathbf{R}^2 are distinct, have no extra preimages than these $|A|$ points, and ϕ has no extra points in \mathbf{R}^2 at which images of three or more different points of S^1 meet.

Obviously, the space of all quasidoodles respecting a given A -configuration J is a linear subspace of codimension $2(|A| - \#A)$ in the space \mathcal{K} of all quasidoodles. The number $|A| - \#A$ is thus called the *complexity* of the configuration. We shall denote this subspace by $\chi(J)$. The set of all quasidoodles, which *strictly* respect this configuration, is an open dense subset in this subspace.

Definition 6. A *degeneration mode* of an A -configuration is some arbitrary order of marking all the points of the configuration, satisfying the following conditions: on any step we mark either some three points of some of $\#A$ groups (if none point of the same group is already marked) or one point of a group, some three or more points of which are already marked.

Any degeneration process of a regular quasidoodle ϕ defines in the obvious way a degeneration mode of the A -configuration strictly respected by ϕ .

Let M be an invariant of doodles, and J an A -configuration.

Proposition 5 (cf. Theorem 5 in [35]). *If M is an invariant of order i , and $|A| - \#A = i$, then for any regular quasidoodle ϕ , which strictly respects the A -configuration J , any characteristic number defined by the triple consisting of M , ϕ and a degeneration process of ϕ , depends only on the pair consisting of the configuration J and its degeneration mode defined by this degeneration process.* \square

Corollary. *For any j the group of order j invariants of doodles or I -doodles is finitely generated.*

Indeed, the number of its generators does not exceed the sum (over all equivalence classes of A -configurations of complexity $\leq i$) of numbers of their degeneration modes. \square

Any invariant of order i can be encoded by its *characteristic table* which we now describe.

This table has $i + 1$ levels numbered by $0, 1, \dots, i$. The l -th level consists of several cells, which are in one-to-one correspondence with all possible pairs consisting of

- a) an equivalence class of A -configurations of complexity l in S^1 ,
- b) a degeneration mode of this A -configuration.

In each cell we indicate

- a) a picture (or a code) representing a “model” regular quasidoodle, which strictly respects some A -configuration from the corresponding equivalence class (this picture is the same for all invariants),
- b) a degeneration process of this quasidoodle, defining this degeneration mode (also not depending on the invariant), and
- c) the characteristic number, which our invariant and the degeneration process, corresponding to the cell, assign to this quasidoodle.

By the Proposition 5, we may not specify the pictures and degeneration processes in the cells of the highest (i -th) level of the table: indeed, the corresponding characteristic numbers depend only on the data indexing the cell.

For instance, the 0-th level consists of the trivial \bigcirc -like doodle, and the corresponding characteristic number equals 0 (we can normalize all invariants so that they take zero value on the trivial doodle). The 1-st level is empty, because there are no configurations of complexity 1.

Having these data, we can calculate our invariant by the inductive process, coinciding identically with the one described in § 3 of [35] or § 4.2 of [32] (where, however, the characteristic numbers sometimes are called "actuality indices", and all (quasi)doodles should be replaced by (quasi)ornaments or (singular) knots).

Remark 2. Of course, the characteristic numbers corresponding to different degeneration modes of the same regular quasidoodle satisfy some natural relations. For instance, if two degeneration modes differ only by a reordering of markings, preserving their order inside any group of the A -configuration, then the corresponding characteristic numbers coincide.

Less trivial identities, relating different degeneration modes inside the same group, follow from the differentials in the chain complex of *connected hypergraphs*, see [34], [12].

3. INVARIANTS OF DOODLES IN TERMS OF THE RESOLVED DISCRIMINANT

We shall work with the space $\mathcal{K} \equiv C^\infty(S^1, \mathbf{R}^2)$ as with an Euclidean space of a very large but finite dimension Δ . The justification of this assumption uses the finite-dimensional approximations of this space and is similar to that given in [35], [32]. A rigorous reader can everywhere below consider \mathcal{K} as a generic finitedimensional subspace in $C^\infty(S^1, \mathbf{R}^2)$.

In particular, we shall use the Alexander duality formula

$$(3.1) \quad \tilde{H}^i(\mathcal{K} \setminus \Sigma) \simeq \bar{H}_{\Delta-1-i}(\Sigma),$$

where \tilde{H}^* is the usual reduced cohomology group (we are especially interested in the group $\tilde{H}^0(\mathcal{K} \setminus \Sigma)$ of invariants taking zero value on the trivial doodle), and \bar{H}_* is the *Borel–Moore homology group*, i.e. the homology group of the one-point compactification reduced modulo the added point.

3.1. Simplicial resolution of the discriminant variety. Denote by Ψ the *configuration space* of all unordered collections of three points in S^1 , so that $\Psi = (S^1)^3/S(3)$. It is a smooth 3-dimensional manifold with corners, homotopy equivalent to S^1 . More precisely, it is the space of an orientable fiber bundle over S^1 , whose projection p sends a triple

of points to their sum in the Lie group S^1 , the fiber is a closed filled triangle, and the monodromy over the entire base S^1 provides the cyclic permutation of sides and vertices of the triangles. The "zero section", consisting of centers of these fibers, consists of configurations, all whose points are at the distance $2\pi/3$ one from the other.

Let us fix a space \mathbf{R}^N of a huge dimension (much greater than that of the space \mathcal{K}) and fix a generic embedding $\lambda : \Psi \rightarrow \mathbf{R}^N$. For any point $\phi \in \Sigma$ consider all such points $\{x, y, z\} \in \Psi$ that one of three conditions holds:

- a) $x \neq y \neq z \neq x$ and $\phi(x) = \phi(y) = \phi(z)$;
- b) $x = z \neq y$ and $\phi'(x) = 0, \phi(x) = \phi(y)$;
- c) $x = y = z$ and $\phi'(x) = \phi''(x) = 0$.

Then consider all points $\lambda(\{x, y, z\}) \in \mathbf{R}^N$ for all such triples $\{x, y, z\}$. Since our embedding is generic (and N is sufficiently large) then for any $\phi \in \Sigma$ the convex hull of all such points is a simplex with vertices at all these points. (Using the *generic* finite-dimensional approximations of the space \mathcal{K} we can ignore the situation when the number of such triples is infinite, moreover, we can assume that the number of such triples has a finite upper estimate, uniform over all $\phi \in \Sigma$.)

Denote this simplex by $\sigma(\phi)$.

Finally, define the *resolution set* $\sigma \subset \mathcal{K} \times \mathbf{R}^N$ as the union of all simplices of the form $\phi \times \sigma(\phi)$ over all $\phi \in \Sigma$.

The obvious projection $\mathcal{K} \times \mathbf{R}^N \rightarrow \mathcal{K}$ provides the map $\pi : \sigma \rightarrow \Sigma$. By definition, this map is surjective, and by the previous "finiteness assumption" it is also proper.

Proposition 6 (cf. [32], [35]). *The map π provides the homotopy equivalence of one-point compactifications of spaces σ and Σ . \square*

In particular, the Borel–Moore homology groups (see (3.1)) of these spaces are canonically isomorphic.

3.2. A -cliques and the main filtration of the resolved discriminant. The space σ admits a natural filtration, which can be defined in two equivalent ways. To do it, we need to extend the notion of an A -configuration used in § 2.2. Again, let A be a finite series of $\#A$ integer numbers $A = (a_1, a_2, \dots, a_{\#A})$, all of which are ≥ 3 .

Definition 7. An A -clique is an unordered collection of $a_1 + \dots + a_{\#A}$ points in S^1 , divided into groups of cardinalities $a_1, \dots, a_{\#A}$, such that a) points of different groups do not coincide geometrically; b) points inside a group can coincide, but with multiplicity at most 3. Again, the *complexity* of a clique J is the number $|A| - \#A$; another important

characteristic, $\rho(J)$, is the number of geometrically distinct points in it, i.e. the dimension of the space of cliques equivalent to it.

The map $\phi : S^1 \rightarrow \mathbf{R}^2$ *respects* an A -clique if it glues together all geometrically distinct points inside any its group, satisfies the condition $\phi' = 0$ at all points of multiplicity 2, and satisfies the condition $\phi' = \phi'' = 0$ at all points of multiplicity 3. ϕ *strictly respects* it, if additionally it does not respect any cliques of larger complexity.

For any A -clique J , consider all triples of points in S^1 belonging to the same its group. All such triples are the points of the configuration space Ψ . Consider the images in \mathbf{R}^N of all these points under the embedding λ and define the simplex $\sigma(J) \subset \mathcal{K} \times \mathbf{R}^N$ as the convex hull of all such points.

Example 1. Suppose that $\#A = 1$, $a_1 = 4$, and the unique group of our A -clique is a quadruple of points (x, x, y, z) , exactly two of which coincide. Then the simplex $\sigma(J)$ is a triangle with 3 vertices (x, x, y) , (x, x, z) , (x, y, z) .

Now, for any natural i we take all quasidoodles, *strictly* respecting all possible A -cliques of complexities $\leq i$, then consider the union of their complete preimages in σ and, finally, define the term σ_i of our *main filtration* as the closure of this union. It contains also some points of the form $\phi \times \theta \in \mathcal{K} \times \mathbf{R}^N$, where ϕ is a quasidoodle of complexity $> i$, and θ is some boundary point of the corresponding simplex $\sigma(\phi)$.

Equivalently, for any A -clique J we can define the linear subspace $\chi(J) \subset \mathcal{K}$ consisting of all quasidoodles respecting (strictly or not) this clique. The term $\sigma_i \subset \sigma$ of the main filtration is then defined as the union of all subsets $\chi(J) \times \sigma(J) \subset \mathcal{K} \times \mathbf{R}^N$ over all cliques J of complexity $\leq i$. It is easy to see that these two definitions of the main filtration are equivalent.

Definition 8. An element of the group $\bar{H}_*(\Sigma) \equiv \bar{H}_*(\sigma)$ is of order i if it can be realized by a locally finite cycle lying in the term σ_i of this filtration. In particular, an invariant of doodles is of order i if its class in the group (3.1) can be realized as a linking number with the direct image of a cycle lying in σ_i .

Proposition 7 (cf. [35], Theorem 7). *The last definition of the order of invariants is equivalent to that given in § 2. \square*

4. CALCULATION OF INVARIANTS OF DOODLES

Consider the spectral sequence $E_{p,q}^r$ calculating the Borel–Moore homology group of the space σ and generated by our filtration. Its term $E_{p,q}^1$ is isomorphic to $\bar{H}_{p+q}(\sigma_p \setminus \sigma_{p-1})$.

Proposition 8 (cf. [35]). *All groups $E_{p,q}^1$ with $p + q \geq \Delta$ are equal to 0.*

The proof of this proposition will be given in § 4.4.

Hence, for the calculation of the group of invariants (by (3.1) coinciding with $\bar{H}_{\Delta-1}(\sigma)$) only the $(\Delta - 1)$ - and $(\Delta - 2)$ -dimensional Borel–Moore homology groups of these spaces $\sigma_i \setminus \sigma_{i-1}$ are interesting.

In this section we calculate these groups for $i \leq 4$.

4.1. Stratification of the resolved discriminant. By construction, any space $\sigma_i \setminus \sigma_{i-1}$ consists of several \mathcal{J} -blocks, numbered by all equivalence classes \mathcal{J} of A -cliques of complexity exactly i . Given such an equivalence class \mathcal{J} , the corresponding block $B(\mathcal{J})$ is the space of a fiber bundle, whose base is the space of all A -cliques J of this class, and the fiber over a clique J is the direct product of

a) a linear subspace of codimension $2i$ in \mathcal{K} , consisting of all quasi-doodles respecting J (the vector bundle of such subspaces is always orientable) and

b) a dense subset of the simplex $\sigma(J)$ (namely, this simplex minus some its faces, which may belong to σ_{i-1} ³).

By the Thom isomorphism, the Borel–Moore homology group \bar{H}_* of any such block is canonically isomorphic to the group $\bar{H}_{*-(\Delta-2i)}$ of the space of only the second bundle of complexes b). Such spaces will be called the *reduced \mathcal{J} -blocks*.

The bases of these fiber bundles are $\rho(\mathcal{J})$ -dimensional manifolds, where $\rho(\mathcal{J})$ is the number of geometrically distinct points in any clique J of the class \mathcal{J} .

4.2. The auxiliary filtration.

Definition 9. We introduce the *auxiliary filtration* in the space $\sigma_i \setminus \sigma_{i-1}$, defining its term F_α as the union of all above-described blocks over all classes \mathcal{J} of A -cliques such that $\rho(\mathcal{J}) \leq \alpha$.

Example 2. The term σ_2 of the main filtration consists of exactly 3 terms $F_1 \subset F_2 \subset F_3 \equiv \sigma_2$ of the auxiliary filtration, because the (3)-cliques $\{x, y, z\}$ can consist of 1, 2 or 3 geometrically different points.

Definition 10. The spectral sequence, calculating the group $\bar{H}_*(\sigma_i \setminus \sigma_{i-1})$ and generated by this auxiliary filtration, is called the *auxiliary spectral sequence* in contrast to the *main* one generated by the main filtration and calculating the homology groups of entire σ .

³If J is an A -configuration, i.e. has no multiple points, then these are exactly the faces corresponding to *not connected* 3-hypergraphs, see [34], [12]

4.3. Columns $p = 1$ and $p = 2$ of the main spectral sequence.
The term σ_1 is empty, because there are no cliques of complexity 1.

Proposition 9. *The column $\{p = 2\}$ of the term E^1 of the main spectral sequence has only the following nontrivial elements: $E_{2,\Delta-5}^1 \simeq \mathbf{Z}$ and $E_{2,\Delta-6}^1 \simeq \mathbf{Z}$.*

Proof. The term σ_2 is the space of a fiber bundle, whose base is the configuration space Ψ , and the fiber over its point $\{x, y, z\}$ is the linear subspace of codimension 4 in \mathcal{K} consisting of all quasidoodles respecting the corresponding (3)-clique. This bundle is orientable, therefore $\bar{H}_*(\sigma_2) = \bar{H}_{*-\Delta+4}(\Psi) \simeq H_{*-\Delta+4}(S^1)$. \square

4.4. Revised resolution. Before continuing, we improve slightly the construction of our \mathcal{J} -blocks, described in the subsection 4.1. Namely, any of these blocks contains a deformation retract having the same Borel–Moore homology group; it will be convenient to consider these new blocks $VB(\mathcal{J})$, which we (in accordance with the terminology of [35]) shall call the *visible blocks*.

Again, any such block, corresponding to an equivalence class \mathcal{J} of A -cliques, is the space of a fibered product of two bundles, whose base and the first factor of the fiber are the same as previously (i.e. respectively the space of all cliques $J \in \mathcal{J}$ and an oriented vector space of dimension $\Delta - 2(|A| - \#A)$). However, the second factors of the fibers of this new bundle are some subcomplexes of the barycentric subdivision of the corresponding fibers of the former bundle of simplices.

Namely, consider any A -clique J and the corresponding simplex $\sigma(J)$. Any vertex of this simplex, i.e. a triple of points of our A -clique lying inside one its group, defines a vector subspace of codimension 4 in \mathcal{K} . Consider all these subspaces corresponding to our clique. (E.g., if the clique is an A -configuration, then there are exactly $\sum_{i=1}^{\#A} \binom{a_i}{3}$ such different subspaces.) These subspaces together with all their possible intersections form a partially ordered set $\Pi(J)$ with respect to the relation of the (inverse) inclusion. This poset has unique maximal element: the subspace $\chi(J)$, i.e. the intersection of all our subspaces.

Having such a partially ordered set, we can define its *order complex* $\diamond(J)$ (see e.g. [11]): this is a formal simplicial complex, whose simplices are all the strictly monotone sequences of elements of our poset.

E.g., if the A -clique J is an A -configuration, then such simplices of the maximal dimension in $\diamond(J)$ are nothing but the degeneration modes of J described in § 2.2.

This order complex $\diamond(J)$ can be naturally considered as a subcomplex of the barycentric subdivision of the simplex $\sigma(J)$ (while its maximal element $\{\chi(J)\}$ corresponds to the center of $\sigma(J)$).

If the complexity $|A| - \#A$ of J is equal to j , then the set $\diamond(J) \cap \sigma_{j-1}$ of "marginal" faces of this complex consists of all its faces not containing the main vertex $\{\chi(J)\}$.

Define the *visible block* $VB(\mathcal{J})$ corresponding to our equivalence class \mathcal{J} of A -cliques as the subset of the previously defined block $B(\mathcal{J})$, described in § 4.1, in which elements of the fibers $\sigma(J)$ should belong to the subcomplex $\diamond(J) \subset \sigma(J)$.

Define the revised resolved discriminant $\diamond \subset \sigma$ as the union of all such revised blocks $VB(\mathcal{J})$. It has a natural *main* filtration $\{\diamond_i\}$ induced by the identical embedding from the main filtration in σ . Similarly, in any space $\diamond_i \setminus \diamond_{i-1}$ the auxiliary filtration is induced from that in $\sigma_i \setminus \sigma_{i-1}$.

Proposition 10 (see [35], [37]). *The inclusion $\diamond \hookrightarrow \sigma$ induces a homotopy equivalence of one-point compactifications of these spaces. The same is true for the inclusion $\diamond_j \hookrightarrow \sigma_j$ of any terms of the main filtrations and also for the inclusions $VB(\mathcal{J}) \hookrightarrow B(\mathcal{J})$ of blocks in them corresponding to the same classes of equivalent A -cliques of complexity j . In particular this inclusion induces an isomorphism of both the main and auxiliary spectral sequences calculating the Borel–Moore homology groups of all these objects. \square*

The Proposition 8 follows immediately from this one and the following lemma.

Lemma 1. *The dimension of any block $VB(\mathcal{J})$ of \diamond does not exceed $\Delta - 1$.*

Proof. This dimension consists of three numbers:

- a) the dimension $\rho(\mathcal{J})$ of the space of cliques of our class (which does not exceed the number $|A|$);
- b) the dimension $\Delta - 2(|A| - \#A)$ of the standard fiber $\chi(J)$ of the first (vector) bundle;
- c) the dimension of the order complex of subspaces associated with the clique J .

The last dimension is equal to the length of the maximal monotone chain of subspaces constituting our poset $\Pi(J)$ minus 1. This length is equal to $|A| - 2\#A$. Indeed, all our subspaces are of even codimension in \mathcal{K} , their maximal element is the space $\chi(J)$ of codimension $2(|A| - \#A)$, and there are exactly $\#A$ steps in the chain, when the dimension jumps

by 4 (when we start a new group of points; the very first step is one of them).

Finally, we obtain that the dimension of our block is not greater than $|A| + \Delta - 2(|A| - \#A) + |A| - 2\#A - 1$. \square

Corollary. Calculating the invariants of doodles, we can consider only the \mathcal{J} -blocks corresponding to such A -cliques J , that $|A| - \rho(A) \leq 1$, i.e. either all their points are geometrically distinct or there is at most one point of multiplicity 2. Moreover, the blocks of the latter kind can provide only relations in the group of invariants $\bar{H}_{\Delta-1}(\sigma)$, but not its generators.

4.5. The third column of the spectral sequence.

Theorem 4. *The group $E_{3,q}^1 = \bar{H}_{3+q}(\diamond_3 \setminus \diamond_2)$ of the main spectral sequence is trivial for all numbers q not equal to $\Delta - 6$ and $\Delta - 7$, while $E_{3,\Delta-6}^1 \simeq \mathbf{Z} \simeq E_{3,\Delta-7}^1$.*

The proof of this theorem occupies the rest of this subsection.

The term $\diamond_3 \setminus \diamond_2$ consists of exactly 4 \mathcal{J} -blocks corresponding to different equivalence classes \mathcal{J} of cliques of complexity 3: the main block A (all 4 points in the cliques are distinct), the block B of auxiliary filtration 3 (exactly two points coincide), and two blocks of auxiliary filtration 2: C (one simple point and one point of multiplicity 3) and D (two double points).

Consider the corresponding reduced blocks (see § 4.1) $[A], [B], [C]$ and $[D]$.

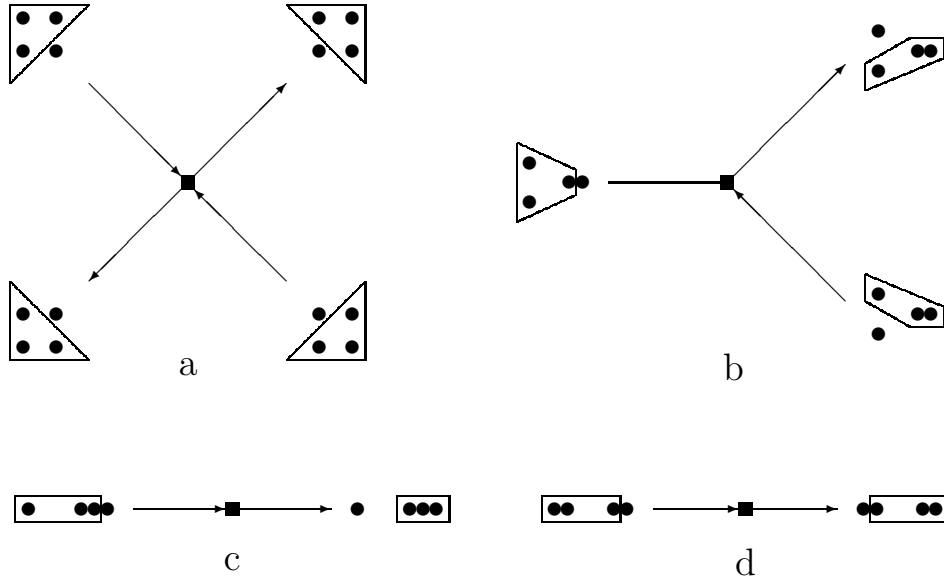
The main reduced block $[A]$ is the space of a fiber bundle, whose base \mathcal{J} is the configuration space $B(S^1, 4)$ of subsets of cardinality 4 in S^1 , and the fiber is a cross with its four endpoints removed. The vertices of such a cross correspond to all possible choices of some 3 points of the 4-point configuration; this notation is transparent in Fig. 9a.

The base $B(S^1, 4)$, in its turn, is the space of the fiber bundle

$$(4.1) \quad p : B(S^1, 4) \rightarrow S^1$$

where the projection sends a quadruple of points in $S^1 = \mathbf{R}/\mathbf{Z}$ to their sum (mod \mathbf{Z}), and the fiber is diffeomorphic to an open 3-dimensional disc. The monodromy over the basis circle of (4.1) violates the orientation of the bundle of 3-dimensional discs and acts on the bundle of crosses as a cyclic permutation of their edges.

Thus the Wang exact sequence of this bundle gives us the following assertion.

FIGURE 9. Order complexes for blocks in \diamond_3

Proposition 11. *The Borel–Moore homology group of the reduced main block $[A]$ of $\diamond_3 \setminus \diamond_2$ is nontrivial only in dimensions 5 and 4 and is isomorphic to \mathbf{Z} in these dimensions. The generator of the 5-dimensional group is swept out by the bundle over $B(S^1, 4)$ of 1-chains shown in Fig. 3b \equiv Fig. 9a (i.e. the unique chains antiinvariant under the monodromy action). The generator of the 4-dimensional group is swept out by the fiber bundle, whose base is the fiber $p^{-1}(0)$ of the bundle (4.1), and fibers are 1-chains shown in these pictures by the sum of two right-hand arrows. \square*

The block $[B]$ also is the space of a (trivial) fiber bundle, whose base is the space of all configurations of 3 points in S^1 , one of which (the double point) is distinguished, and the fiber is a star with 3 rays without endpoints. These fibers and their endpoints are shown in Fig. 9b: two right-hand endpoints correspond to the subspaces in \mathcal{K} given by the conditions of the form $\phi(x) = \phi(y), \phi'(x) = 0$ and $\phi(x) = \phi(z), \phi'(x) = 0$, while the left endpoint corresponds to the equation $\phi(x) = \phi(y) = \phi(z)$, cf. Example in § 3.2.

The base of this bundle is diffeomorphic to the direct product $S^1 \times B^2$, where B^2 is an open 2-dimensional disc. Monodromy along the basic circle S^1 acts trivially on the bundle of 3-stars. Therefore we have the following statement.

	$q \uparrow$				
$\Delta - 5$		\mathbf{Z}^2	\mathbf{Z}^2	\mathbf{Z}	
$\Delta - 6$		\mathbf{Z}^2	\mathbf{Z}^2	\mathbf{Z}	
		2	3	4	$a \rightarrow$

FIGURE 10. Auxiliary spectral sequence for the column $p = 3$.

Proposition 12. *The Borel–Moore homology group of the reduced block $[B]$ is nontrivial only in dimensions 4 and 3 it is isomorphic to \mathbf{Z}^2 in both these dimensions.* \square

In a similar way we get the following statements.

Proposition 13. *The Borel–Moore homology group of the reduced block $[C]$ is nontrivial only in dimensions 3 and 2 and is isomorphic to \mathbf{Z} in both these dimensions.* \square

Proposition 14. *The Borel–Moore homology group of the reduced block $[D]$ is nontrivial only in dimensions 3 and 2 and is isomorphic to \mathbf{Z} in both these dimensions.* \square

The corresponding order complexes are shown in Figs. 9c and 9d, respectively.

Corollary. *The term E^1 of the spectral sequence, calculating the Borel–Moore homology group of the (not reduced) term $\diamond_3 \setminus \diamond_2$ of the main filtration of our resolved discriminant and generated by the auxiliary filtration in this term, looks as is shown in Fig. 10 (i.e., all its cells $E^1_{p,q}$ other than six indicated there are trivial).*

Proposition 15. *The Borel–Moore homology group of the subspace in $\diamond_3 \setminus \diamond_2$ formed only by the blocks A , B and C , is acyclic in all dimensions.*

This follows immediately from the shape of all these blocks and their generators, and from accounting the limit positions of subspaces in \mathcal{K} corresponding to the cliques of types A and B when they degenerate and form configurations of types B and C, respectively.

E.g. let us consider the (4)-clique $::$ drawn at any endpoint of Fig. 9a and let the two right-hand points of it move one towards the other, forming at the last instant a clique of type B. The subspaces of codimension 2 in \mathcal{K} , corresponding to all endpoints of the cross, tend to similar subspaces for endpoints of the star of Fig. 9b. Namely, to both left-hand endpoints of the cross \times there corresponds the unique

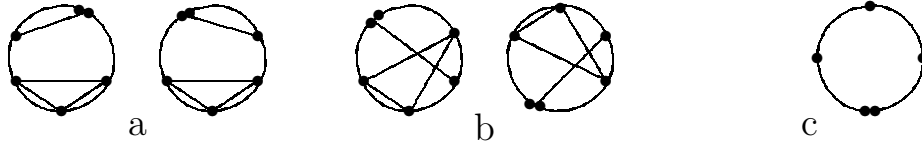


FIGURE 11. Degenerated blocks of complexity 4

left-hand endpoint of the star, and other two endpoints "remain unmoved". Therefore the boundary under this degeneration of the basic cycle shown in Fig. 9a is equal to the basic cycle in $\bar{H}_{\Delta-3}$ (the block B) depicted by two arrows in Fig. 9b (i.e., swept out by the fiber bundle of such 1-chains over entire base $S^1 \times B^2$ of this block).

Other boundary operators $\bar{H}_*(A) \rightarrow \bar{H}_*(B)$ and $\bar{H}_*(B) \rightarrow \bar{H}_*(C)$ can be considered in a similar way and give the assertion of Proposition 15. \square

Corollary. *The Borel–Moore homology group of the space $\diamond_3 \setminus \diamond_2$ coincides with that of unique its block D (described in Proposition 14).*

This terminates the proof of Theorem 4. \square

4.6. Invariants of order 4. In this subsection, we shall be interested only in the $(\Delta - 1)$ -dimensional homology group of the space $\diamond_4 \setminus \diamond_3$, which can provide invariants of doodles.

In accordance with the Corollary of Proposition 10, we shall consider only blocks of complexity 4 having at most one double point.

Proposition 16. *There are exactly 4 \mathcal{J} -blocks of complexity 4 corresponding to classes \mathcal{J} of A -configurations (i.e. of A -cliques, all whose points are geometrically distinct). Three of them correspond to $(3, 3)$ -cliques consisting of 6 points in S^1 separated into two triples in one of ways shown in Fig. 4a. The fourth corresponds to the unique class of (5) -configurations.*

Also, there are exactly 5 blocks of complexity 4 corresponding to cliques with exactly one point of multiplicity 2. Four of them also are of type $A = (3, 3)$ and can be obtained from first two pictures in Fig. 4a by some degenerations, see Figs. 11a and 11b respectively. The fifth \mathcal{J} -block corresponds to (5) -cliques with exactly one double points, see Fig. 11c. \square

Let us study all these \mathcal{J} -blocks and their homology groups.

Proposition 17. *All three \mathcal{J} -blocks corresponding to three pictures of Fig. 4a are smooth orientable manifolds diffeomorphic to $S^1 \times \mathbf{R}^{\Delta-2}$, in*

particular any of them has only two nontrivial Borel–Moore homology groups $\bar{H}_{\Delta-1} \simeq \mathbf{Z} \simeq \bar{H}_{\Delta-2}$. \square

More precisely, in the first and the third (respectively, in the second) case the corresponding space \mathcal{J} of equivalent cliques is diffeomorphic to a nonorientable (respectively, orientable) fiber bundle over S^1 with fiber B^5 . The order complexes $\diamond(J)$ in all cases are homeomorphic to open intervals, whose endpoints correspond to subspaces of codimension 2 in \mathcal{K} defined by the (3)-cliques forming the triangles, and the center corresponds to the subspace of codimension 4, defined by their intersection. The bundle of these intervals is (non)orientable exactly in the same cases when the corresponding configuration space is.

Proposition 18. *The $(\Delta-1)$ -dimensional Borel–Moore homology group of the \mathcal{J} -block in $\diamond_4 \setminus \diamond_3$, corresponding to the unique class \mathcal{J} of (5)-configurations, is equal to \mathbf{Z}^2 .*

(I thank very much A. B. Merkov, who proved this proposition, and also suggested the following notation, convenient for the homological study of such blocks.)

Proof. Let J be a configuration of 5 different points in S^1 . The corresponding order complex $\diamond(J)$ is two-dimensional. Its 20 simplices of dimension 2 are in a natural one-to-one correspondence with the triples of the form {some 3 points of J ; some 4 points containing these three; all five points}. Denote such a simplex by the arrow, connecting two points *not participating* in the first triple and directed towards the point not participating in the quadruple. The one-dimensional simplices of the same order complex $\diamond(J)$ are the segments of the following three kinds.

A) connecting (the vertex corresponding to the subspace in \mathcal{K} defined by) a triple of our points and (the vertex corresponding to its subspace defined by) a quadruple containing this triple. Such edges do not belong to \diamond_4 and are not interesting for us.

B) connecting (the vertex corresponding to) a triple and (that corresponding to) the maximal element $\chi(J) \in \diamond(J)$ (defined by the entire (5)-clique J). These 10 edges are denoted by non-oriented edges connecting two points not participating in the triple.

C) connecting (the vertices corresponding to) a quadruple and $\chi(J)$. These 5 edges are denoted by marking the point not participating in the quadruple.

In this notation, the boundary of an arrow (i.e., a 2-simplex of $\diamond(J)$) is equal to the edge, obtained from this arrow by forgetting the orientation, minus the endpoint of the arrow.

Lemma 2. *The Borel–Moore homology group of this complex is located in dimension 2 and is isomorphic to \mathbf{Z}^6 .*

Proof. Indeed, the generating it 2-cycles look as follows.

First of all, any arrow can appear in such a cycle only together with its opposite, taken with opposite coefficient: otherwise the boundary of this cycle will contain an edge of type B) with a non-zero coefficient. Thus it is sufficient to consider the 10-dimensional group, generated by the linear combinations of the form {an arrow minus its opposite}. Such an element will be depicted by a double arrow directed as the first arrow in this combination, see Fig. 4b. A linear combination of such double arrows is a cycle of the complex of closed chains of $\diamond(J) \setminus \diamond_3$ if and only if the correspondingly oriented segments form a cycle of the complete graph with 5 vertices. The group H_1 of this complete graph is isomorphic to \mathbf{Z}^6 , and lemma is proved. \square

Further, our \mathcal{J} -block is the space of a fiber bundle, whose base is the configuration space $B(S^1, 5) \cong S^1 \times B^4$, and the fiber over the configuration J is the direct product of the oriented $(\Delta - 8)$ -dimensional subspace $\chi(J) \subset \mathcal{K}$ and the complex $\diamond(J) \setminus \diamond_3$. The monodromy over the generator S^1 of the fundamental group of the base acts on this complex (and its homology) as a cyclic permutation of 5 vertices. Thus by the Wang exact sequence the group considered in Proposition 18 is generated exactly by all cycles of a complete 5-graph which are invariant under this action. This group is two-dimensional; its generators are shown in Fig. 4b. \square

We have found all possible generators of the group

$$(4.2) \quad \bar{H}_{\Delta-1}(\diamond_4 \setminus \diamond_3),$$

namely the following statement holds.

Proposition 19. *Any element of the group (4.2) is a linear combination of five chains shown in Fig. 4. \square*

Now let us study the boundaries of these chains in other blocks.

Proposition 20. *Two chains corresponding to two left pictures in Fig. 4a cannot participate in an element of the group (4.2) with nonzero coefficients.*

Indeed, the boundary of the first (respectively, the second) of them contains the sum of generators of $(\Delta - 2)$ -dimensional homology groups of two blocks shown in Fig. 11a (respectively, 11b). These generators do not appear in the boundaries of any other of our 5 chains. \square

Proposition 21. *Two chains corresponding to two pictures in Fig. 4b cannot participate in an element of the group (4.2) with nonzero coefficients.*

To prove this proposition, let us consider the homology group of the \mathcal{J} -block, corresponding to (5)-cliques $J = (x, x, y, z, w)$, as shown in Fig. 11c. The corresponding order complex $\diamond(J)$ again is two-dimensional.

Proposition 22. *For any (5)-clique J consisting of exactly 4 geometrically different points in S^1 , the group $\bar{H}_*(\diamond(J) \setminus \diamond_3)$ is concentrated in dimension 2 and is isomorphic to \mathbf{Z}^3 . The Borel–Moore homology group of the corresponding block in $\diamond_4 \setminus \diamond_3$ is concentrated in dimension $\Delta - 2$ and also is isomorphic to \mathbf{Z}^3 .*

Proof. Exactly as in the proof of Proposition 18, almost all two-dimensional simplices of the complex $\diamond(J)$ are naturally depicted by arrows connecting some of 4 geometrically distinct points of the clique J . (The unique extra triangle is the triple of subspaces, whose first element is defined by three points of multiplicity 1: it should be denoted by a loop edge, connecting the double point with itself. This triangle cannot participate with non-zero coefficient in any cycle of the complex $\diamond(J) \setminus \diamond_3$.)

Again, the cycles of this complex are depicted by double arrows, forming cycles (in the usual sense) of the complete graph on our 4 vertices. This proves the first assertion of Proposition 22.

The entire J -block is the space of a fiber bundle over the space of all cliques of this type (which is diffeomorphic to $S^1 \times B^3$), namely, of a fibered product of an orientable $(\Delta - 8)$ -dimensional vector bundle and the (trivial) bundle of complexes $\diamond(J) \setminus \diamond_3$. This proves the last assertion of Proposition. Moreover, the $(\Delta - 2)$ -dimensional cycles of the block are in a one-to-one (Künneth) correspondence with 2-dimensional cycles of $\diamond(J) \setminus \diamond_3$, where J is any clique of this class. \square

Proposition 23. *The boundaries in this block of two basic $(\Delta - 1)$ -dimensional cycles, shown in Fig. 4b, are the two cycles shown in Fig. 12a. \square*

Proposition 24. *The chain in \diamond_4 , shown in Fig. 1 (\equiv the right-hand picture in Fig. 4a), defines a cycle in $\diamond_4 \setminus \diamond_3$.*

Proof. The $(3, 3)$ -cliques of this type $*$ can degenerate only in the following way: some two neighboring points of *different* groups coincide, thus forming a (5)-configuration, see the left picture of Fig. 12b. Given



FIGURE 12. Boundaries of basic cycles of Fig. 4b and Fig. 1

a (5)-configuration J , the \mathcal{J}' -block of the type $*$ adjoins the corresponding complex $\diamond(J)$ exactly 5 times, because such coincidence can happen at any of its 5 points. The boundary position of complexes $\diamond(J')$, when $J' \in *$ tends to J in the way shown in Fig. 12b left, is (in the notation used in the proof of Proposition 18) equal to the difference of two 1-dimensional simplices in $\diamond(J)$ denoted by two edges in Fig. 12b right. The sum of such differences over all 5 vertices of the configuration J is equal to 0, and Proposition 24 is proved. \square

Remark 3. A much more general fact was proved in [22]: any "horizontal" boundary operator d^1 of the auxiliary spectral sequence, corresponding to the collision of two points of two different groups, always is trivial.

Summarizing the Propositions 16–24, we get the following statement.

Theorem 5. *The group $\bar{H}_{\Delta-1}(\diamond_4 \setminus \diamond_3)$ is isomorphic to \mathbf{Z} and is generated by the fundamental cycle of the \mathcal{J} -block corresponding to the (3,3)-cliques shown in Fig. 1.*

As the $(\Delta - 1)$ - and $(\Delta - 2)$ -dimensional Borel–Moore homology groups of both spaces \diamond_2 and $\diamond_3 \setminus \diamond_2$ are trivial (see Proposition 9 and Theorem 4), this implies Theorem 1 of the Introduction.

4.7. A nontrivial doodle. The theory of finite-order invariants provides a method of constructing a priori nontrivial (and nonequivalent) objects. E.g., imagine that we do not know any nontrivial knot in \mathbf{R}^3 and wish to construct it. To do it, we can calculate the simplest finite-order knot invariant (given by the chord diagram \oplus), then draw the simplest singular knot respecting this diagram (i.e. having two transverse selfintersections), and then consider four knots obtained from it by all possible local resolutions of both these points. At least one of obtained knots surely will be nonequivalent to the others (and indeed, if we do all this in the simplest possible way, we get three trivial knots and one trefoil).

In exactly the same way, we can construct the simplest quasidoodle with two generic triple points, respecting the triangular diagram of

Fig. 1. Perturbing it in four different ways, we shall obtain three trivial (equivalent to a circle) doodles, and one equivalent to Fig. 2a.

(However, A. B. Merkov, who discovered this doodle, came to it from very different considerations.)

5. INVARIANTS OF I-DOODLES

In this and the next sections we consider only the immersed curves in \mathbf{R}^2 . To any immersion $\phi : S^1 \rightarrow \mathbf{R}^2$ there corresponds a map $S^1 \rightarrow S^1$: any point $x \in S^1$ goes to the direction of the tangent vector $\phi'(x)$. Accordingly to S. Smale [29], this correspondence is a homotopy equivalence between spaces $IK \equiv Imm(S^1, \mathbf{R}^2)$ and $C(S^1, S^1)$. These spaces split into countably many components labeled by the "winding numbers" (i.e. the indices of corresponding maps $S^1 \rightarrow S^1$). Any of these components is homotopy equivalent to S^1 , the homotopy equivalence being provided by the image of (the tangent direction of ϕ at) the distinguished point of S^1 .

The discriminant $I\Sigma$ in the space IK is just the intersection of this space IK with the discriminant set $\Sigma \subset \mathcal{K}$ considered in the previous sections. Its resolution $I\Diamond$ is a subset in \Diamond , namely the complete preimage of $I\Sigma$.

In its decomposition into \mathcal{J} -blocks only the \mathcal{J} -configurations, i.e. the \mathcal{J} -cliques without multiple points, can take part.

Proposition 25. *For any connected component \mathcal{C} of the space IK and for any A -configuration J in S^1 , the space of immersions $S^1 \rightarrow \mathbf{R}^2$ respecting this configuration and lying in this component is a path-connected open submanifold of the space $\chi(J) \subset \mathcal{K}$.*

This follows easily from the Smale's theorem. \square

For any component \mathcal{C} , denote by $\mathcal{C}I\Diamond$ and $\mathcal{C}I\Diamond_i$ the intersection of the space $I\Diamond$ (respectively, $I\Diamond_i$) with the preimage of \mathcal{C} under the projection $\Diamond \rightarrow \mathcal{K}$.

Example 3. The stratum $I\Diamond_2$ is an open subset in the space of a fiber bundle, almost coinciding with that considered in Proposition 9, with unique difference that its base is not the entire space $\Psi = (S^1)^3/S(3)$, but its open part $B(S^1, 3)$. The *strangeness* is the linking number in \mathcal{C} with the direct image of the fundamental cycle of this subset. The existence of the strangeness as an integer-valued invariant is due to the fact that this configuration space $B(S^1, 3)$ is orientable.

All other $(\Delta - 1)$ -dimensional blocks in all spaces $I\Diamond_3 \setminus I\Diamond_2$ and $I\Diamond_4 \setminus I\Diamond_3$ are the open subsets of similar blocks considered in the

	$q \uparrow$	0		
$\Delta - 3$	Z	0		
$\Delta - 4$	Z	Z	0	
$\Delta - 5$	0	Z	Z ⁵	
		0		
		2	3	4
		$p \rightarrow$		

FIGURE 13. Spectral sequence for invariants of orders 2, 3 and 4

previous section; the corresponding virtual generators of the group of invariants of I-doodles are shown in Figs. 3b, 4a and 4b.

All these generators define elements of corresponding groups $\bar{H}_{\Delta-1}(\diamond_i \setminus \diamond_{i-1}) \equiv E_{i,\Delta-i-1}^1$. For two generators shown in Fig. 4b this follows from the fact that the entire boundaries of corresponding blocks in $I\diamond_4 \setminus I\diamond_3$ are empty. For two remaining blocks of Fig. 4a we should additionally check that their boundaries in the block corresponding to the (5)-configurations are trivial; the proof of this fact essentially coincides with that of Proposition 24.

So, for any fixed component of the space IK the domain in the table $\{E_{p,q}^1\}$ responsible for the calculation of invariants of orders 2, 3 and 4 of I-doodles from this component looks as is shown in Fig. 13.

Proposition 26. *For any connected component \mathcal{C} of the space IK , the fragment of the spectral sequence shown in Fig. 13 degenerates at the term E^1 , i.e. all its elements extend to well defined Borel–Moore homology classes of the space $CI\diamond$.*

Proof. For the group $E_{2,\Delta-3}$ this is obvious.

The group $E_{2,\Delta-4}^1$ is generated by the fundamental cycle of the submanifold in $CI\diamond_2$, consisting of such pairs of the form $\{\text{a 3-configuration } (x, y, z) \in B(S^1, 3); \text{ a map } \phi : S^1 \rightarrow \mathbf{R}^2\}$ that $x + y + z \equiv 0 \pmod{2\pi}$. It is obviously a cycle in entire $CI\diamond$, let us prove that it is not homologous to zero. As $H^2(\mathcal{C}) \simeq 0$, it is sufficient to construct two 1-dimensional cycles in $\mathcal{C} \setminus I\Sigma$, defining the same element in $H^1(\mathcal{C})$ but such that some (and then any) 2-chain realizing the homology between these cycles has nonzero intersection number with this fundamental cycle.

Consider a map $\phi \in I\Sigma \cap \mathcal{C}$ with unique generic triple point, and let ϕ_1, ϕ_2 be two its small nondiscriminant perturbations resolving this triple point in two different ways, see Fig. 6c.

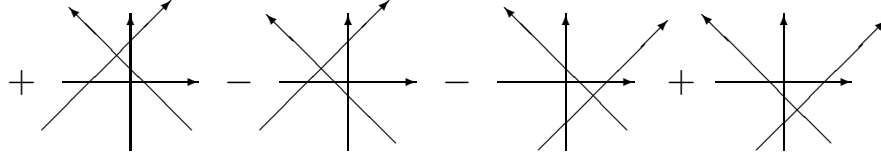


FIGURE 14. The chain of I-doodles nontrivial by the invariant of order 3

For any $i = 1, 2$, denote by $[\phi_i]$ the 1-cycle in $\mathcal{C} \setminus \Sigma$ swept out by all maps obtained from ϕ_i by all cyclic reparametrizations of the issue circle S^1 . These two cycles are obviously homologous in \mathcal{C} , and the intersection index of such a homology with the above manifold is equal to ± 3 .

This proves the assertion of Proposition for cells $E_{2,\Delta-4}$ and $E_{3,\Delta-4}$.

In particular, there exist two elements of the group $H_0(\mathcal{C} \setminus \Sigma)$, i.e. two linear combinations of doodles in \mathcal{C} , which cannot be distinguished by the "strangeness" (generating the group $E_{2,\Delta-3}^\infty$ of second order 0-cohomology classes), but can be distinguished by the invariant generating the group $E_{3,\Delta-4}^\infty$. (In accordance with § 4.7, we can find these combinations by resolving unique point of multiplicity 4. Indeed, the linear combination of four doodles, locally situated as in Fig. 14 and coinciding outside it, provides such a chain.)

Exactly as above we produce from them an 1-cycle in $\mathcal{C} \setminus \Sigma$, homologous to zero in \mathcal{C} , and having a nonzero intersection index with the chain generating the group $E_{3,\Delta-5}^1$. This proves our Proposition. \square

In particular, we have proved that for any component \mathcal{C} of the space IK all 7 generators mentioned in Theorem 2 and shown in Figs. 3, 4 define independent elements of the group $\bar{H}_{\Delta-1}(\Sigma \cap \mathcal{C})$.

Finally, it is obvious that the intersection indices with all these $(\Delta - 1)$ -dimensional Borel–Moore homology classes of the discriminant define zero elements in the 1-dimensional cohomology group of the component \mathcal{C} , and hence the linking numbers with them are well defined invariants of I-doodles.

Theorem 2 is thus completely proved.

6. 1-DIMENSIONAL COHOMOLOGY OF THE SPACE OF IMMERSIONS $S^1 \rightarrow \mathbf{R}^2$ WITHOUT POINTS OF MULTIPLICITY 4

Define the discriminant $I\Sigma 4 \subset IK$ as the set of immersions $\phi : S^1 \rightarrow \mathbf{R}^2$ such that images of some 4 different points coincide.

Its resolution $I\Diamond 4$ is constructed in essentially the same way as it was done above for the set $I\Sigma$. In this section we are interested in the

1-dimensional cohomology classes of the space $IK \setminus I\Sigma 4$ or, which is the same, in the $(\Delta - 2)$ -dimensional Borel–Moore homology classes of spaces $I\Sigma 4$ or $I\Diamond 4$.

The first nonempty term of this resolution is of filtration 3. This is the space of an orientable $(\Delta - 6)$ -dimensional vector bundle over the configuration space $B(S^1, 4)$. This configuration space is non-orientable, therefore the fundamental cycle of this term defines a class only in the group $\bar{H}_{\Delta-2}(I\Sigma 4, \mathbf{Z}_2)$, but not in the integer homology group, see in Theorem 3 the statement " $\mathcal{F}_3 \sim \mathbf{Z}_2$ over \mathbf{Z}_2 ".

The next term $I\Diamond 4_4 \setminus I\Diamond 4_3$ of our filtration also is the space of a fiber bundle, whose base is the configuration space $B(S^1, 5)$, and the fibers are direct products of stars \star with 5 rays (without endpoints, which belong to the smaller term of the filtration) and some (canonically oriented) $(\Delta - 8)$ -dimensional vector subspaces in \mathcal{K} .

The base $B(S^1, 5)$ of this bundle is orientable (and diffeomorphic to $S^1 \times \mathbf{R}^4$), and the monodromy over the circle generating the group $\pi_1(B(S^1, 5)) \sim \mathbf{Z}$ acts on the fibration of 5-stars \star by cyclic permutations of their rays.

Therefore the $(\Delta - 2)$ -dimensional Borel–Moore homology group of this term coincides with the subgroup of the group $\bar{H}_1(\star)$ consisting of elements invariant under the rotations of these stars. For any coefficient group G this group is isomorphic to G^4 . If in G the condition $5a = 0$ implies $a = 0$, then its invariant subgroup is trivial; in the case $G = \mathbf{Z}_5$ this group is isomorphic to \mathbf{Z}_5 , see statement " $\mathcal{F}_4 \simeq \mathbf{Z}_5$ " over \mathbf{Z}_5 of Theorem 3.

Finally, consider the term $I\Diamond 4_5 \setminus I\Diamond 4_4$ of our filtration. It is the space of a fiber bundle over $B(S^1, 6)$, whose fiber is the product of $\mathbf{R}^{\Delta-10}$ and some two-dimensional order complex. This complex is similar to the one considered in the proof of Proposition 18, with unique difference: its vertices correspond to choices of some 4, 5 or 6 points of our 6, and not of 3, 4 or 5 points of 5. Absolutely as previously, the two-dimensional cycles of this complex are the linear combinations of (double) arrows with starts and ends at these 6 points, forming the cycles (in the usual sense) of the complete graph on these 6 vertices.

However, unlike the case of 5-configurations, the base space $B(S^1, 6)$ is non-orientable. Therefore the $(\Delta - 2)$ -dimensional Borel–Moore homology classes of our block are in a one-to-one correspondence with such cycles of the complete 6-graph, which are *anti-invariant* under the cyclic permutations of its 6 vertices.

The group of such cycles can be easily calculated and is isomorphic to \mathbf{Z}^2 ; the pictures of its generators are given in Fig. 5.

By dimensional reasons, these homology classes of the space $I\Diamond_5 \setminus I\Diamond_4$ can be extended to these of the space $I\Diamond_5$, and hence to the 1-dimensional cohomology classes (of order 5) of the entire space of immersions $S^1 \rightarrow \mathbf{R}^2$ without 4-fold points.

This proves Theorem 3 of the Introduction.

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ON FINITE ORDER INVARIANTS OF TRIPLE POINTS FREE PLANE CURVES ~~3~~

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